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# Oscillated densely packed granular media immersed in a fluid

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**Abstract**

A densely packed bed of granular material immersed in a fluid is cyclically agitated. The fluid may be compressible due to a small amount of gas trapped in the pores. From the general two-phase equations the oscillatory motion characteristics are determined. A penetration depth of the order of magnitude of a few particle diameters is found. Secular cycle-averaged effects are investigated and two are identified: a particle pressure due to irreversible processes and collisions during the oscillation, which is linear in the velocity amplitude and a quadratic effect that arises due to non-zero correlations between the fluctuations in solids volume fraction and those in velocity, displacement and pore pressure. Orders of magnitude and time constants of both these secular effects are established. The analysis shows that the secular effects manifest themselves in a few cycles of the vibration. From constitutive estimates the influence on mean intergranular stress and interstitial pore pressure is obtained. It is found that the quadratic effect hardly affects the intergranular stress, but has a substantial influence on the pore pressure, which is reduced. The linear particle pressure effectively decreases the magnitude of the compressive intergranular stress. Linear and quadratic effects have markedly different frequency dependencies. A comparison with experiment on this point is reported. Applications are vibrated filtration and agitated magma chambers that contain sediment. For the latter the theory explains the formation of gas bubbles associated with rapid fluid pressure reduction during earthquake loading.


## 0. Introduction

This paper is concerned with the mechanics of a rapidly agitated, densely packed, fluid-immersed particulate aggregate. There are two key aspects to this problem. The first is the behaviour of the oscillated bed, which deals with the absorption of energy, penetration depth and modes of motion. The second is the so-called *secular* part of the problem, that is the slow, quasi-static motion and development of stresses and pressures, which are associated with the oscillatory motion. For example, a densely packed, fluid-immersed bed which is oscillated from below may exhibit fluidisation phenomena. In this paper both aspects will be treated. There are a number of practical applications for this problem in civil and chemical engineering and in geology. Oscillated-septum dead-end filtration—a chemical engineering application—has been studied in the past [1–3] (in the dead-end filter geometry there is also a mean downward flow, which exerts a force on the particle bed).

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**Nomenclature**

| | |
|---|---|
| $a$ | mean particle radius (m) |
| $A$ | Eq. (26) (dimensionless) |
| $B$ | Eq. (26) (dimensionless) |
| $c$ | proportionality constant for the particle pressure (dimensionless) |
| $C_E$ | rate of skeletal stiffness reduction (Pa) |
| $C_R$ | rate of fluid drag resistance reduction (Pa sm$^{-2}$) |
| $d$ | displacement (m) |
| $D_E$ | rate of change of $E_f$ (dimensionless) |
| $D_R$ | $C_R a^2/\eta$ (dimensionless) |
| $e$ | $\frac{1-\phi}{\phi}$ (dimensionless) |
| $e_{zz}$ | excess strain (dimensionless) |
| $E$ | skeletal stiffness (Pa) |
| $E_f$ | $(2.17-e)^2/(1+e)$ (dimensionless) |
| $g$ | acceleration due to gravity (m s$^{-2}$) |
| $h$ | gap width (m) |
| $H$ | layer thickness (m) |
| $p$ | fluid pressure (Pa) |
| $q$ | $(1-\phi)u$ (m s$^{-1}$) |
| $r$ | $\phi v$ (m s$^{-1}$) |
| $R(\phi)$ | solidosity dependent drag coefficient (Pa s m$^{-2}$) |
| $R$ | fluid drag resistance (Pa s m$^{-2}$) |
| $t$ | time (s) |
| $u$ | fluid velocity (m s$^{-1}$) |
| $v$ | solids velocity (m s$^{-1}$) |
| $\mathbf{x}$ | position vector (m) |
| $z$ | vertical position (m) |

*Greek letters*

| | |
|---|---|
| $\beta$ | compressibility (Pa$^{-1}$) |
| $\gamma$ | damping coefficient (Pa s) |
| $\delta$ | Kronecker delta (dimensionless) |
| $\eta$ | melt viscosity (Pa s) |
| $\kappa$ | coefficient for the resistance $\overline{R}$ (dimensionless) |
| $\lambda$ | inverse penetration depth (m$^{-1}$) |
| $\lambda_0$ | $Re(\lambda)=\sqrt{\overline{R}\overline{\phi}\omega\zeta/(2\overline{E})}/(1-\overline{\phi})$ (dimensionless) |
| $\phi$ | solids volume fraction (dimensionless) |
| $\pi$ | particle pressure (Pa) |
| $\chi$ | attached fluid mass coefficient (dimensionless) |
| $\rho_f$ | melt density (kg m$^{-3}$) |
| $\rho_s$ | solids density (kg m$^{-3}$) |
| $\Sigma$ | partial solid-phase stress (Pa) |
| $\sigma$ | partial fluid-phase stress (Pa) |
| $\tau$ | intergranular stress (Pa) |
| $\omega$ | circular frequency (Hz) |
| $\varsigma$ | $\overline{E}\beta(1-\overline{\phi})+1$ (dimensionless) |

The purpose of the agitation in this case is to clear the septum of particulates so unclogging the filter, which enhances the performance. The geophysical application is quite novel and concerns earthquakes in volcano chambers. It is believed that over time, as the volcano is dormant, a granular deposit develops in the chamber. When the latter is agitated by an earthquake the pore pressure is rapidly reduced in value, thus allowing gas bubbles to form. This explains the origin of the gas [4,5]. Clearly, modelling this phenomenon is crucial for the understanding of the combination of parameters that influence the likelihood of volcanic activity. There is ample descriptive literature on the subject [6–8], but hardly any modelling of this multi-physics problem.

The filter problem and the volcano problem operate in quite different parameter ranges. For example, the viscosity of the fluid in the filter problem is typically that of water (0.001 Pa s), while magma viscosity may be three or four orders of magnitude larger. Water, furthermore, may be regarded as incompressible, while magma with a small gas content is essentially compressible. As approximations will be introduced to lead to transparent results these ranges are important. The comparison between the parameter ranges for the dead-end filtration problem and the agitated magma chamber problem is very interesting, as it transpires that though frequencies and viscosities are very different, the penetration depth turns out to be roughly the same. Comparing these two cases, therefore, highlights two limits of the manner in which slow secular effects become manifest.

In this paper the general theory of oscillated particle–fluid beds is briefly reviewed. Then the secular effects are discussed, especially the particle pressure (which is a linear effect) and the quadratic secular terms. The oscillated bed is then calculated and approximations that are appropriate to the parameter ranges are introduced, leading to a simple expression for the penetration depth. The associated secular contributions to the static stresses are then estimated. The time-development of the latter is shown to take place in the dynamic range (that is, a time constant of the order of magnitude of the period of oscillation is obtained). Finally, the range of influence of the secular effects on both the particle-phase and fluid-phase stress is estimated and discussed in light of experimental results.

## 1. Oscillated packed beds and slurries (general theoretical considerations)

### 1.1. Governing equations

The particle-phase mass density is denoted by $\rho_s$; the solids volume fraction is $\phi$ and the solid phase velocity is $\mathbf{v}$. The fluid mass density is called $\rho_f$ and the fluid-phase velocity $\mathbf{u}$. All these are field variables and depend on the position $\mathbf{x}$ and the time $t$. The basic continuity equations for the solid phase and fluid phase are given in the literature, see Refs. [9–11]:

$$\frac{\partial(\rho_s\phi)}{\partial t}+\frac{\partial(\rho_s v_i\phi)}{\partial x_i}=0, \tag{1}$$

$$\frac{\partial}{\partial t}[\rho_f(1-\phi)]+\frac{\partial[(1-\phi)\rho_f u_i]}{\partial x_i}=0, \tag{2}$$

where Einstein's summation convention is used.

The solids mass density is assumed to be constant, but the fluid may be compressible and therefore $\rho_f$ can depend on position and time. The main cause of the compressibility is the presence of a small amount of gas in bubble form. The relation between fluid compressibility $\beta$ and fluid pressure $p$ is a constitutive relation of the form $\rho_f^{-1}\overline{\mathrm{D}}\rho_f/\overline{\mathrm{D}}t=\beta\overline{\mathrm{D}}p/\overline{\mathrm{D}}t$ (the time derivative $\overline{\mathrm{D}}/\overline{\mathrm{D}}t$ is co-moving with the mean fluid motion). The variations in fluid density $\delta\rho_f$ are expected to be small compared to the mean $\overline{\rho}_f$. Eq. (2) is expanded and terms of order $\delta\rho_f/\overline{\rho}_f$ may be neglected compared to unity. Therefore, the continuity equation for the fluid is approximately

$$(1-\phi)\beta\frac{\overline{\mathrm{D}}p}{\overline{\mathrm{D}}t}-\frac{\partial\phi}{\partial t}+\frac{\partial[(1-\phi)u_i]}{\partial x_i}=0. \tag{3}$$

The gas phase is a small minority species embedded in the fluid. No separate equation set for this phase is necessary, as the motion of the gas phase does not significantly affect the stresses in the medium. The gas phase comes in the form of small bubbles.

The equations of motion (the momentum balance equations) for a fluid–solid mixture are well-researched and reported in the literature [9–11]. For each Cartesian component ($\ell=1,2,3$), the equations read

$$\phi\rho_s\frac{\mathrm{D}'v_\ell}{\mathrm{D}t}=\frac{\partial}{\partial x_i}(\phi\Sigma_{\ell i})+\phi R(\phi)(u_\ell-v_\ell)+\chi\phi\rho_f\frac{\mathrm{D}}{\mathrm{D}t}(u_\ell-v_\ell)+\phi\rho_s g_\ell, \tag{4}$$

$$(1-\phi)\rho_f\frac{Du_\ell}{\mathrm{D}t}=\frac{\partial}{\partial x_i}[(1-\phi)\sigma_{\ell i}]-\phi R(\phi)(u_\ell-v_\ell)-\chi\phi\rho_f\frac{\mathrm{D}}{\mathrm{D}t}(u_\ell-v_\ell)+(1-\phi)\rho_f g_\ell. \tag{5}$$

The symbols used in these two vector equations are as follows. In addition to the fluid co-moving derivative $\mathrm{D}/\mathrm{D}t$, a co-moving derivative with the mean solids motion is formally required and is denoted by $\mathrm{D}'/\mathrm{D}t$. The fluid-phase stress is $\boldsymbol{\sigma}$ and the particle-phase stress is $\boldsymbol{\Sigma}$. For the fluid stress a simple isotropic form is assumed: $-p\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the unity tensor (the Kronecker delta). The intergranular stress $\boldsymbol{\tau}$ is obtained from the particle-phase stress and the fluid pressure: $\tau=\phi(\boldsymbol{\Sigma}+p\boldsymbol{\delta})$. The fluid drag force is proportional to the velocity difference in particle and fluid phases with proportionality coefficient $R(\phi)$. A set of terms is introduced to compensate for the attached mass of fluid to the particles. These terms require a proportionality constant $\chi$, which is a phenomenological parameter (a typical value is $\chi=0.5$). Body forces associated with gravity are also introduced; the acceleration due to gravity is $\mathbf{g}$.

There are a number of further small corrective terms reported in the literature, see for example Ref. [11]. These are all neglected for the problem in hand.

In order to distinguish between oscillatory and secular motion the four balance equations are recast in terms of the flux fields $\mathbf{q} \equiv (1-\phi)\mathbf{u}$ and $\mathbf{r} \equiv \phi\mathbf{v}$. This gives

$$\frac{\partial \phi}{\partial t} + \frac{\partial r_i}{\partial x_i} = 0, \tag{6a}$$

$$(1-\phi)\beta \frac{\overline{\mathrm{D}}p}{\overline{\mathrm{D}}t} - \frac{\partial \phi}{\partial t} + \frac{\partial q_i}{\partial x_i} = 0, \tag{6b}$$

$$\rho_s \frac{\mathrm{D}' r_\ell}{\mathrm{D}t} - \frac{\rho_s r_\ell}{\phi}\frac{\mathrm{D}'\phi}{\mathrm{D}t} = \frac{\partial}{\partial x_i}(\phi \Sigma_{\ell i}) + R(\phi)\left(\frac{\phi q_\ell}{1-\phi} - r_\ell\right) + \chi\phi\rho_f \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{q_\ell}{1-\phi} - \frac{r_\ell}{\phi}\right) + \phi\rho_s g_\ell, \tag{6c}$$

$$\rho_f \frac{\mathrm{D}q_\ell}{\mathrm{D}t} + \frac{\rho_f q_\ell}{1-\phi}\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial}{\partial x_i}[(1-\phi)\sigma_{\ell i}] - R(\phi)\left(\frac{\phi q_\ell}{1-\phi} - r_\ell\right) - \chi\phi\rho_f \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{q_\ell}{1-\phi} - \frac{r_\ell}{\phi}\right) + (1-\phi)\rho_f g_\ell. \tag{6d}$$

In the physics of an agitated slurry two time scales need to be distinguished: the rapid oscillatory effect and the accompanying slow, secular (consolidatory or quasi-static) motion. The latter are caused by the excitations of the particles in the slurry, which lead to a *particle pressure* that is more or less constant in time. Both rapid and secular phenomena are described by the above equation set. Oscillatory (sinusoidal) solutions are obtained by solving the equations for the first Fourier component. These give an impression of the decay of the amplitude of oscillation for an agitated packed bed layer. The equilibrium equations for the quasi-static deformation are also derived. The terms in the latter require input from the oscillatory solution—notably an expression for the particle pressure and an evaluation of the time averages of products of oscillating quantities. All solutions are carried out in one dimension—the vertical $z$ direction. The acceleration due to gravity is $g_3 = -g$; $g$ is positive.

In all this constitutive relations for the solid-phase intergranular stress have to be provided. For contacting particles a stiffness $E(\phi)$ is introduced, as well as a damping associated with relative interparticle motion in a fluid. The damping coefficient is called $\gamma$. Strictly speaking the particles could find themselves in a fluidised state; in that case the stiffness becomes zero, but the damping remains. In addition to these traditionally well-known constitutive properties, a particle pressure needs to be introduced to account for the momentum transfer of non-reversibly interacting particles in an agitated slurry.

### 1.2. Particle pressure and distinguishing time scales

The concept of (expansive) particle pressure in a dense slurry was introduced by McTigue and Jenkins [12], who used it to describe migration effects in non-uniformly sheared slurries. A particle pressure arises when there is the possibility of net momentum transfer due to an asymmetry in the particle–particle interaction, in other words if approaching particles and departing particles do not sense equal-magnitude (opposite sign) forces. If a purely fluid-mediated interaction between perfectly smooth particles is considered there is no net transfer, but if particles can touch and the collision occurs at non-zero restitution then a particle pressure develops. The effect is not necessarily confined to the collisional regime. Nonlinearity in the stress–strain response of a packed bed at small compressive stress may also lead to particle pressure type phenomena. This has been investigated recently by Davis and Koenders [13] and to demonstrate the phenomenon a cartoon from that paper is included here: Fig. 1. Basically, the stress–strain curve goes through a hysteresis loop, but does so in such a way that the response of the medium as a whole can be described by an oscillatory part, while necessarily a mean stress must be included, because the mean stress over the loop does not equal the average applied stress. It must be emphasised that the particle pressure, so obtained, is necessarily only a small fraction of the total intergranular stress and for the remainder of this paper this effect is neglected.

If particles make and break contacts at speed substantial transfer of momentum takes place. For oscillated slurries this effect is described by Gundogdu et al. [2] and its order of magnitude is estimated. The fluid interaction for rough particles is calculated using various approaches by Patir and Cheng [14] and Smart and Leighton [15] and—more recently—by Jenkins and Koenders [16]. The particle pressure $\overline{\pi}$ for a

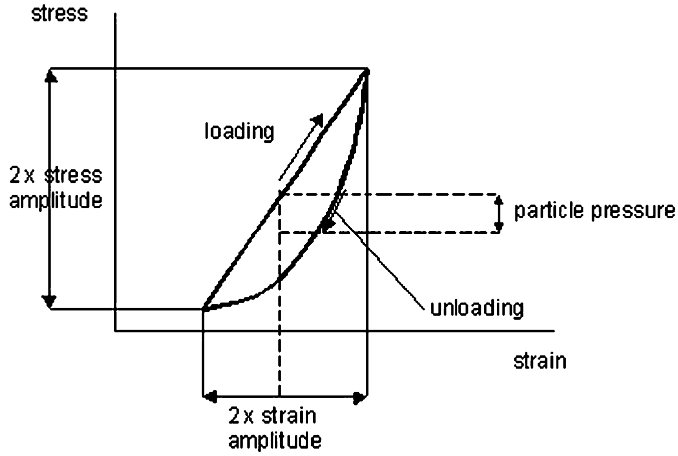


Fig. 1. A strain-controlled cycle of an oscillated packed bed in a nonlinear medium, exhibiting hysteresis. During loading and unloading the stresses follow different paths. The time-average of the stress value $1/T\int_0^T \sigma(t)\,\mathrm{d}t$ over the cycle is not equal to $\frac{1}{2}(\sigma_{\max}-\sigma_{\min})$. The difference is the particle pressure.

granular assembly in a fluid in which the solids oscillate rapidly with a velocity gradient amplitude $|\partial\tilde{v}/\partial z|$ is estimated as

$$\overline{\pi} = c\eta\left|\frac{\partial\tilde{v}}{\partial z}\right|\frac{a}{h}, \tag{7}$$

where $\eta$ is the interstitial fluid viscosity, $a$ the particle radius, $h$ the mean surface to surface distance between the particles that take part in collisions and $c$ a coefficient of order of magnitude 0.1–0.8.

In addition to the particle pressure the equations of motion possess products of oscillating variables. When these are averaged over the cycle time, and depending on the phase shift between the variables, non-zero contributions result. These are quadratic terms in the applied oscillating fields and below an explicit evaluation will be carried out.

The equations that describe the phenomena at the two time scales are obtained as follows. All field parameters are written as a time-averaged part (denoted by an over bar) plus an oscillating part (indicated by a tilde); for example $\phi(\mathbf{x},t)=\overline{\phi}(\mathbf{x},t)+\tilde{\phi}(\mathbf{x},t)$. Then the Eqs. (6a)–(6d) are considered. To begin with the first Fourier component is isolated to give an expression for the oscillating fields. Then the equations themselves are time-averaged to give the slow variation in the process; in addition the dynamic terms are all neglected. The quasi-static evolution in the slurry is a variation superimposed on static equilibrium. It is caused by the 'switching on' of the particle pressure and the average of the quadratic products of the oscillating fields. So the secular effect of the oscillation is that the interstitial fluid pressure and solid-phase stress are modified.

The scenario is informed by the time constants of the problem. Time-averaging is done over periods that are long compared to the period of oscillation, but short compared with the duration of the phenomenon (in the case of the earthquake-excited magma chamber the typical period of oscillation is 0.1 s, while the duration is some 60 s). The secular equations themselves possess a time constant, which will be estimated.

## 2. Packed beds and slurries (estimates of field parameters)

### 2.1. Static limit

First, the purely static limit is considered. All the velocities are zero and all time-dependent terms are irrelevant. The stresses depend on the vertical coordinate $z$ only and they obey the set

$$\frac{\partial\overline{\tau}_{\mathrm{stat}}}{\partial z} - \overline{\phi}\frac{\partial\overline{p}_{\mathrm{stat}}}{\partial z} - \overline{\phi}\rho_s g = 0, \tag{8}$$

$$-\frac{\partial \overline{p}_{\text{stat}}}{\partial z} - \rho_f g = 0. \tag{9}$$

These equations are of course easily solved. At the top of the bed ($z = H$) the intergranular stress and the fluid pressure vanish and therefore

$$\overline{\tau}_{\text{stat}} = \overline{\phi}(\rho_s - \rho_f)g(z - H), \tag{10}$$

$$\overline{p}_{\text{stat}} = -\rho_f g(z - H). \tag{11}$$

For convenience it is here assumed that the fluid and particle layers have the same height. If these are different the formulae are easily adapted. Similarly, if a mean flow is present, the pressure and stress will change, see Ref. [1].

### 2.2. Oscillatory motion

The oscillatory motion is obtained from expanding Eqs. (6a)–(6d) to retain the first Fourier component only. These equations are then solved. All the field parameters depend on position and time according to $e^{-\lambda z}e^{i\omega t}$. Here $\lambda$ is the (complex) wavenumber and $\omega$ the circular frequency of excitation. The interesting solution for the purposes to hand is the one that is localised around the bottom of the bed at $z = 0$; at this point the aggregate is agitated by a sinusoidal displacement $d$ with amplitude $\widehat{d}$.

A key assumption that will be made is that the fluctuations in the solidosity are small compared to the mean value. The equations of continuity and the equations of motion for this first Fourier component become

$$\frac{\partial \widetilde{\phi}}{\partial t} + \frac{\partial \widetilde{r}_i}{\partial x_i} = 0, \tag{12a}$$

$$(1 - \overline{\phi})\beta \frac{\overline{\mathrm{D}}\widetilde{p}}{\overline{\mathrm{D}}t} - \frac{\partial \widetilde{\phi}}{\partial t} + \frac{\partial \widetilde{q}_i}{\partial x_i} = 0, \tag{12b}$$

$$\begin{aligned} \rho_s \frac{\mathrm{D}'\widetilde{r}_\ell}{\mathrm{D}t} - \frac{\rho_s \overline{r}_\ell}{\overline{\phi}} \frac{\mathrm{D}'\widetilde{\phi}}{\mathrm{D}t} = {} & \frac{\partial}{\partial x_i}(\widetilde{\tau}_{\ell i} - \widetilde{\phi p}\delta_{\ell i}) + \overline{R(\phi)}\left(\frac{\overline{\phi}\widetilde{q}_\ell}{1 - \overline{\phi}} - \widetilde{r}_\ell + \frac{\widetilde{\phi}\overline{q}_\ell}{(1 - \overline{\phi})^2}\right) \\ & + \widetilde{R(\phi)}\left(\frac{\overline{\phi}\overline{q}_\ell}{1 - \overline{\phi}} - \overline{r}_\ell\right) + \chi\overline{\phi}\rho_f \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\overline{\phi}\widetilde{q}_\ell}{1 - \overline{\phi}} - \widetilde{r}_\ell + \frac{\widetilde{\phi}\overline{q}_\ell}{(1 - \overline{\phi})^2}\right) + \widetilde{\phi}\rho_s g_\ell, \end{aligned} \tag{12c}$$

$$\begin{aligned} \rho_f \frac{\mathrm{D}\widetilde{q}_\ell}{\mathrm{D}t} + \frac{\rho_f \overline{q}_\ell}{1 - \overline{\phi}} \frac{\mathrm{D}\widetilde{\phi}}{\mathrm{D}t} = {} & -\frac{\partial}{\partial x_\ell}[(\widetilde{1 - \phi})p] - \overline{R(\phi)}\left(\frac{\overline{\phi}\widetilde{q}_\ell}{1 - \overline{\phi}} - \widetilde{r}_\ell + \frac{\widetilde{\phi}\overline{q}_\ell}{(1 - \overline{\phi})^2}\right) \\ & - \widetilde{R(\phi)}\left(\frac{\overline{\phi}\overline{q}_\ell}{1 - \overline{\phi}} - \overline{r}_\ell\right) - \chi\overline{\phi}\rho_f \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\overline{\phi}\widetilde{q}_\ell}{1 - \overline{\phi}} - \widetilde{r}_\ell + \frac{\widetilde{\phi}\overline{q}_\ell}{(1 - \overline{\phi})^2}\right) - \widetilde{\phi}\rho_f g_\ell. \end{aligned} \tag{12d}$$

Interest is now focussed on one-dimensional oscillatory motion in the $z$-direction. The mean motions $\overline{q}$ and $\overline{r}$ are supposed to be small compared to the amplitude of these field variables. Furthermore, the oscillatory parts of the solid phase intergranular stress satisfy the constitutive equation $\widetilde{\tau}_{zz} = \overline{E(\phi)}\widetilde{e}_{zz} + \overline{\gamma(\phi)}\partial\widetilde{v}_z/\partial z + \widetilde{E(\phi)}\overline{e}_{zz} + \widetilde{\gamma(\phi)}\partial\overline{v}_z/\partial z$, where $e_{zz}$ is the strain, $E$ the stiffness and $\gamma$ the damping. For the parameter $\widetilde{R(\phi)}$ a constitutive form $\widetilde{R(\phi)} = C_R\widetilde{\phi}$ is introduced and similarly $\widetilde{E(\phi)} = C_E\widetilde{\phi}$; below it is shown that the damping does not play a prominent role and therefore this parameter is not expanded. Using these elements Eqs. (12a)–(12d) take the form

$$\frac{\partial \widetilde{\phi}}{\partial t} + \frac{\partial \widetilde{r}_z}{\partial z} = 0, \tag{13a}$$

$$(1-\overline{\phi})\beta\frac{\mathrm{D}\widetilde{p}}{\mathrm{D}t}-\frac{\partial\widetilde{\phi}}{\partial t}+\frac{\partial\widetilde{q}_z}{\partial z}=0, \tag{13b}$$

$$\begin{aligned}\rho_s\frac{\partial\widetilde{r}_z}{\partial t}= & \frac{\partial}{\partial z}(\overline{E(\phi)}\widetilde{e}_{zz}+\widetilde{E(\phi)}\overline{e}_{zz})+\frac{1}{\overline{\phi}}\widetilde{\gamma(\phi)}\frac{\partial\overline{r}_z}{\partial z}+\overline{\gamma(\phi)}\frac{\partial}{\partial z}\left(\frac{\widetilde{r}_z}{\overline{\phi}}-\frac{\overline{r}_z\widetilde{\phi}}{\overline{\phi}^2}\right)\\ & -\overline{\phi}\widetilde{p}-\overline{p}\widetilde{\phi}+\overline{R(\phi)}\left(\frac{\overline{\phi}\widetilde{q}_z}{1-\overline{\phi}}-\widetilde{r}_z+\frac{\widetilde{\phi}\overline{q}_z}{(1-\overline{\phi})^2}\right)\\ & +C_R\widetilde{\phi}\left(\frac{\overline{\phi}\overline{q}_z}{1-\overline{\phi}}-\overline{r}_z\right)+\chi\overline{\phi}\rho_f\frac{\partial}{\partial t}\left(\frac{\overline{\phi}\widetilde{q}_z}{1-\overline{\phi}}-\widetilde{r}_z+\frac{\widetilde{\phi}\overline{q}_z}{(1-\overline{\phi})^2}\right)-\widetilde{\phi}\rho_s g,\end{aligned} \tag{13c}$$

$$\begin{aligned}\rho_f\frac{\partial\widetilde{q}_z}{\partial t}= & -\frac{\partial}{\partial z}[(1-\overline{\phi})\widetilde{p}-\widetilde{\phi}\overline{p}]-\overline{R(\phi)}\left(\frac{\overline{\phi}\widetilde{q}_z}{1-\overline{\phi}}-\widetilde{r}_z+\frac{\widetilde{\phi}\overline{q}_z}{(1-\overline{\phi})^2}\right)\\ & -C_R\widetilde{\phi}\left(\frac{\overline{\phi}\overline{q}_z}{1-\overline{\phi}}-\overline{r}_z\right)-\chi\overline{\phi}\rho_f\frac{\partial}{\partial t}\left(\frac{\overline{\phi}\widetilde{q}_z}{1-\overline{\phi}}-\widetilde{r}_z+\frac{\widetilde{\phi}\overline{q}_z}{(1-\overline{\phi})^2}\right)+\widetilde{\phi}\rho_f g.\end{aligned} \tag{13d}$$

These equations are developed for the case when both the constitutive parameters $\overline{E}$, $\overline{\gamma}$, etc. and the mean field variables $\overline{\phi}$, $\overline{p}$ may be considered to be time independent. Now, if the penetration depth is much smaller than the length scale over which the constitutive parameters and mean field variables change appreciably, the average value of the parameters may be used for Eqs. (13a)–(13d). In that case the solution of the oscillatory field parameters has the form $\exp(\lambda z)\exp(\mathrm{i}\omega t)$ and a value for $\lambda$ is easily derived. The average is evaluated over a length scale of order of magnitude of $|\lambda|^{-1}$.

The presence of the fluid pressure presents a problem. However, the context here is a strongly localised phenomenon for which $|\lambda|^{-1}$ is of the order of magnitude of a few mean particle diameters. Let the phenomenon take place in the vicinity of $z=0$, then the pressure field may be replaced by its mean in this small region $\langle\overline{p}\rangle$. The stress divergence terms in the equations are now $\partial/\partial z(\overline{E}\widetilde{e}_{zz}+\overline{\gamma}\partial\widetilde{v}_z/\partial z-\overline{\phi}\widetilde{p}-\overline{p}\widetilde{\phi})=\overline{E}\partial\widetilde{e}_{zz}/\partial z+\overline{\gamma}\partial^2\widetilde{v}_z/\partial z^2-\overline{\phi}\partial\widetilde{p}/\partial z-\langle\overline{p}\rangle\partial\widetilde{\phi}/\partial z$ and $\partial/\partial z[(1-\overline{\phi})\widetilde{p}-\widetilde{\phi}\overline{p}]=(1-\overline{\phi})\partial\widetilde{p}/\partial z-\langle\overline{p}\rangle\partial\widetilde{\phi}/\partial z$. So, it follows that—even considering constant constitutive parameters—the set of equations (13a)–(13d) can in principle only be solved in conjunction with the secular equations, which should yield $\langle\overline{p}\rangle$ and possibly the gradients of this parameter. However, it will be shown that in practice these parameters are not important in the estimate of $\lambda$.

The equation set (13a)–(13d) is a linear system, with a characteristic polynomial $P(\lambda)$. The roots of the equation $P(\lambda)=0$ yield a set of solutions for $\lambda$. In order to obtain the polynomial $P(\lambda)$ set $\widetilde{e}_{zz}=(\mathrm{i}\omega)^{-1}\partial(\widetilde{r}_z/\overline{\phi}-\overline{r}_z\widetilde{\phi}/\overline{\phi}^2)/\partial z$. It is found that the resulting expression is rather long, but substantial shortening is possible when simplifying assumptions are made. The first of these is:

(1) The mean velocities (and their gradients) are so small that they do not influence the oscillatory behaviour. The reason is that the mean velocities represent secular changes and these take place on a much slower time scale than the oscillatory motion. The polynomial becomes

$$\begin{aligned}P(\lambda)= & -\frac{\lambda^4}{\overline{\phi}}(\overline{E}(1-\overline{\phi})+\mathrm{i}\omega\overline{\gamma}(1-\overline{\phi})+\langle\overline{p}\rangle\overline{\phi})+\lambda^3 g\overline{\phi}(\overline{\phi}(\rho_f-\rho_s)+\rho_s)\\ & +\lambda^2\omega\Bigg[-\frac{\chi\omega\rho_f-\overline{R}}{(1-\overline{\phi})}-\overline{\phi}\omega(\overline{E}\beta\chi\rho_f+\beta\rho_f(\mathrm{i}\omega\gamma\chi-\langle\overline{p}\rangle)+\rho_f-\rho_s)\\ & -\frac{\beta\rho_f\omega(E+\mathrm{i}\omega\gamma)}{\overline{\phi}}+\overline{E}\beta(\mathrm{i}\overline{R}+\rho_f\omega)-\omega(\overline{R}\beta\gamma+\beta\rho_f(\langle\overline{p}\rangle-\mathrm{i}\omega\gamma)-\chi\rho_f+\rho_s)\Bigg]\\ & +\lambda\beta g\omega\{\mathrm{i}\overline{R}\overline{\phi}(\rho_f-\rho_s)-\rho_f\omega[\chi\overline{\phi}^2(\rho_f-\rho_s)+\rho_s(\overline{\phi}-1)]\}\\ & -\beta\omega^3\{\mathrm{i}\overline{R}(\overline{\phi}(\rho_f-\rho_s)-\rho_f)-\rho_f\omega[\chi\overline{\phi}(\overline{\phi}(\rho_f-\rho_s)-\rho_f)+\rho_s(\overline{\phi}-1)]\}.\end{aligned} \tag{14}$$

Table 1
Typical values for the parameters of the problem for a magma chamber

| | | | | | |
|---|---|---|---|---|---|
| $\rho_f$ | $2.3 \times 10^3\,\mathrm{kg\,m^{-3}}$ | $\omega$ | $60\,\mathrm{s^{-1}}$ | $H$ | 100 m |
| $\rho_s$ | $2.5 \times 10^3\,\mathrm{kg\,m^{-3}}$ | $E$ | $1 \times 10^8$ Pa | $\gamma$ | $10\eta$ |
| $\overline{\phi}$ | 0.6 | $\beta$ | $1 \times 10^{-8}$ Pa | $a$ | $10^{-3}$ m |
| $\chi$ | 0.5 | $\eta$ | $1 \times 10$ Pa s | $g$ | $10\,\mathrm{m\,s^{-2}}$ |
| $\overline{R}$ | $10^{10}\,\mathrm{Pa\,s\,m^{-2}}$ | $\kappa$ | 100 | $a/h$ | 100 |
| $\langle p\rangle$ | $10^5$ Pa | $\lambda_0$ | $1/(20 \times a)$ | $\zeta$ | 1.5 |

Table 2
Typical values for the parameters of the problem for an oscillated filter problem (see Ref. [1])

| | | | | | |
|---|---|---|---|---|---|
| $\rho_f$ | $1 \times 10^3\,\mathrm{kg\,m^{-3}}$ | $\omega$ | $600\,\mathrm{s^{-1}}$ | $H$ | 0.1 m |
| $\rho_s$ | $1.6 \times 10^3\,\mathrm{kg\,m^{-3}}$ | $E$ | $1 \times 10^4$ Pa | $\gamma$ | $10\eta$ |
| $\overline{\phi}$ | 0.6 | $\beta$ | $1 \times 10^{-8}$ Pa | $a$ | $10^{-3}$ m |
| $\chi$ | 0.5 | $\eta$ | $1 \times 10^{-3}$ Pa s | $g$ | $10\,\mathrm{m\,s^{-2}}$ |
| $\overline{R}$ | $10^6\,\mathrm{Pa\,s\,m^{-2}}$ | $\kappa$ | 100 | $a/h$ | 100 |
| $\langle p\rangle$ | $10^3$ Pa | $\lambda_0$ | $1/(20 \times a)$ | $\zeta$ | 1 |

This simplifies under certain assumptions informed by typical values for the parameters, listed in Tables 1 and 2. In addition to assumption 1 they are the following:

(2) The stiffness $\overline{E}$ is much greater than the mean fluid pressure $\langle\overline{p}\rangle$. This can be determined simply from the numbers. There may be problems with this when the bed fluidises, which needs separate investigation.

(3) The mean solidosity $\overline{\phi}$ is in the range 0.5–0.7. These are normally expected values for an unfluidised granular medium.

(4) The parameter $\omega\rho_f \ll \overline{R}$. The resistance $\overline{R}$ is of the form $\kappa\eta/a^2$, where $\kappa$ is a factor, $\eta$ the fluid viscosity and $a$ the particle diameter. Now, the ratio $\omega\rho_f/\overline{R}$ is of the form $(\omega a)\rho_f a/(\kappa\eta)$. As $(\omega a)$ has the dimension of a speed, the ratio $\omega\rho_f/\overline{R}$ is akin to a Reynolds number. So, other than doing the numbers for a particular case, it is observed that the oscillatory motion leads to slow interstitial flow.

(5) $g\rho_f$ is much smaller than $|\lambda|E$; this follows simply from a comparison of the numbers for $|\lambda|\sim a^{-1}$. This is an approximation that says something about the nature of the localised phenomenon. Let the fluid pressure be influenced by the oscillatory motion to such an extreme extent that $|\langle\partial\overline{p}/\partial z\rangle| \simeq |\lambda|\langle\overline{p}\rangle$, which is the maximum attenuation of the mean fluid pressure that is possible. Then it follows from assumption 2 that $|\langle\partial\overline{p}/\partial z\rangle| \ll |\lambda|E$, thus extending the finding to the gradient of the pressure.

Using these assumptions the polynomial reduces to

$$P(\lambda) = \frac{\lambda^4}{\overline{\phi}}[(\overline{E} + \mathrm{i}\omega\gamma)(\overline{\phi} - 1)] + \lambda^2\omega[\mathrm{i}\overline{E}\beta\overline{R}\overline{\phi}(\overline{\phi} - 1) - \overline{R}\overline{\phi}(\beta(\overline{\phi} - 1)\gamma\omega + \mathrm{i})] + \mathrm{i}\overline{R}\beta\overline{\phi}\omega^3(-\overline{\phi}(\rho_f - \rho_s) + \rho_f) = 0. \tag{15}$$

This equation is easily solved, though the result is not very transparent. However, an approximation can be made that is relevant to the current application. In practical terms the difference between $\rho_f$ and $\rho_s$ is small, so one may set $\rho_s = \rho_f + \mathrm{d}\rho$, where $\mathrm{d}\rho/\rho_f$ is a number that is rather smaller than unity. It is also observed that in practice $\omega\gamma/\overline{E}$ is a small parameter. Now, making good use of approximation 4, above, it is found that up to first order in $\mathrm{d}\rho$ and $\omega\gamma$ the two roots $\lambda^2$ are

$$\lambda^2 = \frac{\mathrm{i}\overline{R}\overline{\phi}\omega\zeta}{\overline{E}(1-\overline{\phi})^2} + \frac{\mathrm{d}\rho(1-\overline{\phi})\omega^2\beta}{\zeta} + \frac{\gamma\omega\overline{R}\overline{\phi}}{\overline{E}^2(1-\overline{\phi})^2}, -\frac{\beta\mathrm{d}\rho\overline{\phi}\omega^2(1-\overline{\phi})}{\zeta}, \tag{16}$$

where a convenient parameter $\zeta \equiv \overline{E}\beta(1-\overline{\phi}) + 1$ has been introduced.

The largest value (by far) is the first root. For this case, the relationship between the various field parameters is expressed in the ratios of the amplitudes, which are denoted by hats

$$\widehat{p} = -\frac{\sqrt{2\overline{E}\overline{R}}\zeta(1-\mathrm{i})}{2\sqrt{\overline{\phi}\omega}(1-\overline{\phi})}\widehat{r}, \quad \widehat{\phi} = \frac{\sqrt{2\overline{R}\overline{\phi}}\zeta(1-\mathrm{i})}{2\sqrt{\overline{E}\omega}(1-\overline{\phi})}\widehat{r}, \quad \widehat{q} = -\frac{\zeta+\overline{\phi}-1}{\overline{\phi}}\widehat{r}. \tag{17}$$

### 2.3. Cycle-averaged cross-products and particle pressure

Defining the inverse penetration depth $Re(\lambda) = \lambda_0 \equiv \sqrt{\overline{R}\overline{\phi}\omega\zeta/(2\overline{E})}/(1-\overline{\phi})$, the cross product averages that are necessary for the development of the secular equation can be evaluated. For example,

$$\overline{\widetilde{r}_z\widetilde{\phi}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} Re(\widehat{r}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{-\lambda z})Re(\widehat{\phi}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{-\lambda z})\,\mathrm{d}t = \frac{\sqrt{2\overline{R}\zeta\overline{\phi}}\mathrm{e}^{-2\lambda_0 z}}{4\sqrt{\overline{E}\omega}(1-\overline{\phi})}\widehat{r}^2. \tag{18}$$

The others are obtained in a similar way

$$\overline{\widetilde{q}_z\widetilde{\phi}} = -\frac{\sqrt{2\overline{R}\overline{\phi}^3}\zeta(\zeta+\overline{\phi}-1)\mathrm{e}^{-2\lambda_0 z}}{4\sqrt{\overline{E}\omega\overline{\phi}}(1-\overline{\phi})}\widehat{r}^2, \tag{19}$$

$$\overline{\widetilde{p}\widetilde{\phi}} = -\frac{\overline{R}\zeta\mathrm{e}^{-2\lambda_0 z}}{2(1-\overline{\phi})^2}\widehat{r}^2, \tag{20}$$

$$\overline{\frac{\partial\widetilde{p}}{\partial t}\widetilde{\phi}} = 0, \tag{21}$$

$$\overline{\frac{\partial\widetilde{d}}{\partial z}\widetilde{\phi}} = -\frac{\overline{R}\zeta\mathrm{e}^{-2\lambda_0 z}}{2\overline{E}\omega(1-\overline{\phi})^2}\widehat{r}^2. \tag{22}$$

Finally, the particle pressure becomes

$$\overline{\pi} = c\eta\left|\frac{\partial\widetilde{v}}{\partial z}\right|\frac{a}{h} = c\eta\lambda_0|\widehat{r}|\mathrm{e}^{-\lambda_0 z}. \tag{23}$$

### 2.4. Secular motion

The secular equations are obtained from Eqs. (6a)–(6d). They involve averages of fields and averages of double products of fluctuations of fields. Two parameters are introduced, $C_R$ defined as $\partial\overline{R}/\partial\overline{\phi}$ and $C_E$ and $\partial\overline{E}/\partial\overline{\phi}$; straightforward evaluation yields the following:

$$\frac{\partial\overline{\phi}}{\partial t} + \frac{\partial\overline{r}_z}{\partial z} = 0, \tag{24a}$$

$$(1-\overline{\phi})\beta\left(\frac{\partial\overline{p}}{\partial t} + \frac{\overline{q}_z}{1-\phi}\frac{\partial\overline{p}}{\partial z}\right) - \beta\overline{\widetilde{\phi}\frac{\partial\widetilde{p}}{\partial t}} - \frac{\partial\overline{\phi}}{\partial t} + \frac{\partial\overline{q}_z}{\partial z} = 0, \tag{24b}$$

$$\frac{\partial}{\partial z}(\overline{\tau}_{zz} - \overline{\pi} + C_E\overline{\frac{\partial\widetilde{d}}{\partial z}\widetilde{\phi}} - \overline{\phi}\overline{p} - \overline{\widetilde{\phi}\widetilde{p}}) + \overline{R}\left(\frac{\overline{\phi}\overline{q}_z}{(1-\overline{\phi})} - \overline{r}_z\right) + C_R\left(\frac{\overline{\phi}(\overline{\widetilde{\phi}\widetilde{q}_z})}{1-\overline{\phi}} - (\overline{\widetilde{\phi}\widetilde{r}_z}) + \frac{(\overline{\widetilde{\phi}^2})\overline{q}_z}{(1-\overline{\phi})^2}\right) - \overline{\phi}\rho_s g = 0, \tag{24c}$$

$$-\frac{\partial}{\partial z}[(1-\overline{\phi})\overline{p}]+\frac{\partial}{\partial z}(\overline{\widetilde{\phi}\widetilde{p}})-\overline{R}\left(\frac{\overline{\phi}\overline{q}_z}{(1-\overline{\phi})}-\overline{r}_z\right)-C_R\left(\frac{\overline{\phi}(\overline{\widetilde{\phi}\widetilde{q}_z})}{1-\overline{\phi}}-(\overline{\widetilde{\phi}\widetilde{r}_z})+\frac{(\overline{\widetilde{\phi}^2})\overline{q}_z}{(1-\overline{\phi})^2}\right)-(1-\overline{\phi})\rho_f g=0. \tag{24d}$$

The appropriate term for the particle pressure has also been introduced here. The term $\partial\overline{\phi}/\partial t$ is eliminated and the cross products are inserted. These steps lead to the following simple equations:

$$\frac{\partial\overline{\phi}}{\partial t}+\frac{\partial\overline{r}_z}{\partial z}=0, \tag{25a}$$

$$(1-\overline{\phi})\beta\left(\frac{\partial\overline{p}}{\partial t}+\frac{\overline{q}_z}{1-\phi}\frac{\partial\overline{p}}{\partial z}\right)+\frac{\partial\overline{r}_z}{\partial z}+\frac{\partial\overline{q}_z}{\partial z}=0, \tag{25b}$$

$$\frac{\partial}{\partial z}(\overline{\tau}_{zz}-\widehat{\pi}\mathrm{e}^{-\lambda_0 z}-B\mathrm{e}^{-2\lambda_0 z}-\overline{\phi}\overline{p})+\overline{R}\left(\frac{\overline{\phi}\overline{q}_z}{(1-\overline{\phi})}-\overline{r}_z\right)-\overline{\phi}\rho_s g=A\mathrm{e}^{-2\lambda_0 z}, \tag{25c}$$

$$\frac{\partial}{\partial z}[(1-\overline{\phi})\overline{p}]+\overline{R}\left(\frac{\overline{\phi}\overline{q}_z}{(1-\overline{\phi})}-\overline{r}_z\right)-(1-\overline{\phi})\rho_f g=A\mathrm{e}^{-2\lambda_0 z}, \tag{25d}$$

where

$$A=\frac{\zeta(C_R(1-\overline{\phi})+2\overline{R})}{4(1-\overline{\phi})^3}\sqrt{\frac{2\overline{R}\overline{\phi}\zeta}{\omega\overline{E}}}\widehat{r}^2,\quad B=\frac{C_E\overline{R}\zeta}{2\overline{E}\omega(1-\overline{\phi})^2}\widehat{r}^2,\quad \widehat{\pi}=c\eta\lambda_0|\widehat{r}|,\ \ \widehat{r}=\omega\overline{\phi}\widehat{d}. \tag{26}$$

Some time after the oscillation has been 'switched on' the system will be in steady state: the velocities and time derivatives will be negligible. Then, assuming $H\gg\lambda_0^{-1}$, the solution for the stresses is simply

$$\overline{\tau}_{zz}=\widehat{\pi}\mathrm{e}^{-\lambda_0 z}+B\mathrm{e}^{-2\lambda_0 z}-\frac{A\mathrm{e}^{-2\lambda_0 z}}{2\lambda_0(1-\overline{\phi})}+g\overline{\phi}(z-H)(\rho_s-\rho_f), \tag{27}$$

$$\overline{p}=-\frac{A\mathrm{e}^{-2\lambda_0 z}}{2\lambda_0(1-\overline{\phi})}-g(z-H)\rho_f. \tag{28}$$

It is seen that the effect of the oscillation is to reduce the fluid pressure in a thin zone at $z=0$. The compressive stress is increased by the term proportional to $A$, but decreased by the terms proportional to $B$ and $\widehat{\pi}$.

### 2.5. *Time-dependent behaviour at the onset of the oscillation*

The way in which the equilibrium state described by Eqs. (27) and (28) comes into being is now investigated. This is a non-trivial problem. As a first approximation the particle pressure is neglected and the equations are linearised by neglecting products of parameters. The parameters $A$ and $B$ are regarded as functions of time and the purpose of this section is to investigate what functional time dependence the secular equations will impose. The only nonlinearity appears in Eq. (25b), so this equation is approximated as

$$(1-\overline{\phi})\beta\frac{\partial\overline{p}}{\partial t}+\frac{\partial\overline{r}_z}{\partial z}+\frac{\partial\overline{q}_z}{\partial z}=0. \tag{29}$$

Adding the two equations of motion gives

$$\frac{\partial}{\partial z}(\overline{\tau}_{zz}-B\mathrm{e}^{-2\lambda_0 z}-\overline{p})-\overline{\phi}\rho_s g+(1-\overline{\phi})\rho_f g=0, \tag{30}$$

or

$$\overline{\tau}_{zz}-B\mathrm{e}^{-2\lambda_0 z}-\overline{p}=(\overline{\phi}\rho_s g-(1-\overline{\phi})\rho_f g)(z-H). \tag{31}$$

Using the simple constitutive equation

$$\frac{\partial \overline{\tau}_{zz}}{\partial t} = \frac{E}{\overline{\phi}} \frac{\partial \overline{r}}{\partial z}, \tag{32}$$

it follows that

$$\frac{E}{\overline{\phi}} \int_0^t \frac{\partial \overline{r}_z}{\partial z} \, \mathrm{d}t = B\mathrm{e}^{-2\lambda_0 z} + \overline{p}^+, \tag{33}$$

where $\overline{p}^+ = \overline{p} - p_{\text{stat}}$. Differentiate with respect to time

$$\frac{E}{\phi} \frac{\partial \overline{r}_z}{\partial z} = \frac{\partial}{\partial t}(B\mathrm{e}^{-2\lambda_0 z}) + \frac{\partial \overline{p}^+}{\partial t} = \frac{\partial}{\partial t}(B\mathrm{e}^{-2\lambda_0 z}) + \frac{\partial \overline{p}^+}{\partial t}, \tag{34}$$

so,

$$\frac{E}{\phi} \frac{\partial \overline{r}_z}{\partial z} = \frac{\partial}{\partial t}(B\mathrm{e}^{-2\lambda_0 z}) - \frac{1}{(1-\overline{\phi})\beta}\left(\frac{\partial \overline{r}_z}{\partial z} + \frac{\partial \overline{q}_z}{\partial z}\right). \tag{35}$$

Now use Eq. (25d) and differentiate once with respect to $z$:

$$(1-\overline{\phi})\frac{\partial^2 \overline{p}^+}{\partial z^2} + \overline{R}\left(\frac{\overline{\phi}}{(1-\overline{\phi})}\frac{\partial \overline{q}_z}{\partial z} - \frac{\partial \overline{r}_z}{\partial z}\right) - \frac{\partial}{\partial z}(A\mathrm{e}^{-2\lambda_0 z}) = 0. \tag{36}$$

Here are two equations in $\partial \overline{q}_z/\partial z$ and $\partial \overline{r}_z/\partial z$ and the solution is

$$\begin{aligned}\frac{\partial \overline{q}_z}{\partial z} = \frac{1-\overline{\phi}}{\overline{R\phi}(E\beta(1-\overline{\phi})+1)}\Bigg[&(E\beta(1-\overline{\phi})+\overline{\phi})\frac{\partial A(z,t)}{\partial z} \\ &+(1-\overline{\phi})\left(\overline{R\phi}\beta\frac{\partial B(z,t)}{\partial t} - (E\beta(1-\overline{\phi})+\overline{\phi})\frac{\partial^2 \overline{p}^+}{\partial z^2}\right)\Bigg],\end{aligned} \tag{37}$$

$$\frac{\partial \overline{r}_z}{\partial z} = -\frac{1-\overline{\phi}}{\overline{R\phi}(E\beta(1-\overline{\phi})+1)}\left[\frac{\partial A(z,t)}{\partial z} - \overline{R\phi}\beta\frac{\partial B(z,t)}{\partial t} - (1-\overline{\phi})\frac{\partial^2 \overline{p}^+}{\partial z^2}\right]. \tag{38}$$

These are substituted back into Eq. (29) to give

$$-E(1-\overline{\phi})\frac{\partial A(z,t)}{\partial z} - \overline{R\phi}\frac{\partial B(z,t)}{\partial t} + E(1-\overline{\phi})^2\frac{\partial^2 \overline{p}^+}{\partial z^2} - \overline{R\phi}(E\beta(1-\overline{\phi})+1)\frac{\partial \overline{p}^+}{\partial t} = 0. \tag{39}$$

Both $A(z,t)$ and $B(z,t)$ possess a spatial dependence, given by the exponential decay and so it is reasonable to assume that $\overline{p}^+$ also possesses this same dependence. Thus, a purely time-dependent equation is obtained by setting $A(z,t) = A_0(t)\mathrm{e}^{-2\lambda_0 z}$; $B(z,t) = B_0(t)\mathrm{e}^{-2\lambda_0 z}$ and $p^+(z,t) = p_0(t)\mathrm{e}^{-2\lambda_0 z}$; it follows that

$$2\lambda_0 E(1-\overline{\phi})A_0(t) - \overline{R\phi}\frac{\partial B_0(t)}{\partial t} + 4\lambda_0^2 E(1-\overline{\phi})^2 p_0(t) - \overline{R\phi}(E\beta(1-\overline{\phi})+1)\frac{\partial p_0(t)}{\partial t} = 0. \tag{40}$$

Laplace transforming, using the initial conditions $B_0(0)$ and $p_0(0) = 0$ reveals immediately that the equation is exponentially unstable

$$2\lambda_0 E(1-\overline{\phi})\widehat{A}_0(s) - s\overline{R\phi}\widehat{B}_0(s) + 4\lambda_0^2 E(1-\overline{\phi})^2\widehat{p}_0(s) - s\overline{R\phi}(E\beta(1-\overline{\phi})+1)\widehat{p}_0(s) = 0. \tag{41}$$

The instability can be removed if

$$2\lambda_0 E(1-\overline{\phi})\widehat{A}_0(s) - s\overline{R\phi}\widehat{B}_0(s) = f(s)[4\lambda_0^2 E(1-\overline{\phi})^2 - s\overline{R\phi}(E\beta(1-\overline{\phi})+1)], \tag{42}$$

in which case

$$\widehat{p}_0(s) = -f(s), \tag{43}$$

where $f(s)$ is a stable function. Thus, physically, the secular equation imposes a form on the time dependence of $A(z,t)$ and $B(z,t)$.

A realisation of this form is now put forward.
Introduce the function

$$\begin{aligned} F(t) &= -F_0 \mathrm{e}^{t/t_0} \quad \text{if } 0 < t \leqslant T_0 \\ &= -F_0 \mathrm{e}^{T_0/t_0} \quad \text{if } t > T_0. \end{aligned} \tag{44}$$

This function has Laplace transform

$$\widehat{F}(s) = -F_0 \frac{\mathrm{e}^{T_0(1-st_0)/t_0}}{s(1-st_0)} + \frac{1}{s(1-st_0)}. \tag{45}$$

This is the form chosen for $f(s)$. The instability is removed when

$$t_0 = \frac{\overline{R\phi}(E\beta(1-\overline{\phi})+1)}{4\lambda_0^2 E(1-\overline{\phi})^2}. \tag{46}$$

To find the corresponding forms for $A(z,t)$ and $B(z,t)$ consider the function

$$\begin{aligned} G(t) &= 0 \quad \text{if } 0 < t \leqslant q_0 \\ &= \frac{t-q_0}{\theta_0} \quad \text{if } q_0 < t \leqslant Q_0 \\ &= \frac{Q_0 - q_0}{\theta_0} \quad \text{if } t > Q_0. \end{aligned} \tag{47}$$

This function has Laplace transform

$$\widehat{G}(s) = \frac{1}{s^2\theta_0}(\mathrm{e}^{-q_0 s} - \mathrm{e}^{-Q_0 s}). \tag{48}$$

The functions $A_0(t)$ and $B_0(t)$ will have the form of $G(t)$ with coefficients $q_A$, $Q_A$ and $\theta_A$ and $q_B$, $Q_B$ and $\theta_B$ instead of $q_0$, $Q_0$ and $\theta_0$. The form $C_1\widehat{A} - sC_2\widehat{B}$ is then

$$C_1\widehat{A} - sC_2\widehat{B} = -\frac{C_1\mathrm{e}^{-(Q_A-q_A)s-q_As}}{s^2\theta_A} + \frac{C_2\mathrm{e}^{-(Q_B-q_B)s-q_Bs}}{s\theta_B} + \frac{C_1\mathrm{e}^{-q_As}}{s^2\theta_A} - \frac{C_2\mathrm{e}^{-q_Bs}}{s\theta_B}. \tag{49}$$

Choose $q_B = 0$ and let $Q_A - q_A$ be vanishingly small, then

$$C_1\widehat{A} - sC_2\widehat{B} = \frac{C_1(Q_A-q_A)\mathrm{e}^{-q_As}}{s\theta_A} + \frac{C_2\mathrm{e}^{-Q_Bs}}{s\theta_B} - \frac{C_2}{s\theta_B}. \tag{50}$$

Finally, choose $q_A = Q_B$; this gives

$$C_1\widehat{A} - sC_2\widehat{B} = \mathrm{e}^{-q_As}\left(\frac{C_1(Q_A-q_A)}{s\theta_A} + \frac{C_2}{s\theta_B}\right) - \frac{C_2}{s\theta_B}. \tag{51}$$

All these are used in Eq. (41) with $C_1 = 2\lambda_0 E(1-\overline{\phi})$, $C_2 = \overline{R\phi}$, resulting in

$$\mathrm{e}^{-q_As}\left(\frac{C_1(Q_A-q_A)}{s\theta_A} + \frac{C_2}{s\theta_B}\right) - \frac{C_2}{s\theta_B} - 4\lambda_0^2 E(1-\overline{\phi})^2 F_0\left(\frac{\mathrm{e}^{T_0(1-st_0)/t_0}}{s} - \frac{1}{s}\right) = 0 \tag{52}$$

and the various parameters are determined:

$$\theta_B F_0 = \frac{\overline{R\phi}}{4\lambda_0^2 E(1-\overline{\phi})^2}, \quad q_A = T_0, \quad \frac{\theta_A}{Q_A - q_A} F_0 = -\frac{\overline{R\phi}}{2\lambda_0(1-\mathrm{e}^{T_0/t_0})}. \tag{53}$$

The parameters $T_0$ and $F_0$ are determined from

$$\frac{Q_A - q_A}{\theta_A} = A, \quad \frac{Q_B}{\theta_B} = B, \tag{54}$$

where $A$ and $B$ are defined in Eq. (26).

The outcome of this analysis is then that in order to achieve stability the parameter $B_0(t)$ climbs linearly with time-constant $\theta_B$ for a time $T_0$; after that $B_0(t)$ remains constant at $B$. At the point $t = T_0$ the parameter $A_0(t)$ starts up; this parameter practically jumps to its final value $A$. The pressure meanwhile rises exponentially with a time constant $t_0$ to its new equilibrium value and reaches that value in a time $T_0$. After $t = T_0$ the pressure remains constant for as long as the oscillation phenomenon lasts at its equilibrium value given by Eq. (28). To give an impression of the numbers for the earthquake agitated magma chamber, using the values in Table 1, $t_0 \simeq 0.01$ s and $T_0 \simeq 0.25$ s. These time values should be seen in the context of the period of oscillation of 0.1 s, so that the equilibrium value is attained after about three cycles. The value of the time constant $t_0$ suggests that the treatment of the time-dependence of the secular equations should include dynamic terms as well. The conclusion may be drawn that the secular effects come into being in a short period of time (order of magnitude of the cycle period). The duration of the whole phenomenon is estimated at some 60 s, so the validity of the analysis is not in doubt.

## 3. Results for the static secular fields (parameter estimates)

The deviation from the static stress due to the agitated bed depends on the numerical range of the parameters. Crucial among these is the value of the drag $\overline{R}(\overline{\phi})$ and its derivative $C_R(\overline{\phi})$. Various formulae are put forward in the literature, see for example Ref. [17]. Here the one proposed by Happel and Brenner [18] is used; it depends on the melt viscosity $\eta$ and the particle radius $a$:

$$\overline{R}(\overline{\phi}) = \frac{9\eta}{2a^2} \frac{\overline{\phi}(1-\overline{\phi})(3+2\overline{\phi}^{5/3})}{3-\frac{9}{2}\overline{\phi}^{1/3}+\frac{9}{2}\overline{\phi}^{5/3}-3\overline{\phi}^2}. \quad (55)$$

For convenience introduce the non-dimensional parameters $R_f \equiv Ra^2/\eta$ and $D_R \equiv C_R a^2/\eta$.

For the rheology of the dynamic stiffness $E(\overline{\phi})$ the literature on oscillated compressed packed beds is useful. According to Hardin and Richart [19] this modulus (in samples with isotropic prestress $\tau_{\rm iso}$) depends on the void ratio $e(=(1-\overline{\phi})/\overline{\phi})$ and has the form $E = f_0 \tau_{\rm iso}^{1/2}(2.17-e)^2/(1+e)$, where $f_0$ depends on the material parameters of the grain. The non-dimensional parameters $E_f \equiv (2.17-e)^2/(1+e)$ and $D_E \equiv \partial E_f/\partial\overline{\phi}$ are introduced. All the $\overline{\phi}$-dependent parameters are plotted in Fig. 2. Note that $C_E$ is rather greater than $E$ itself and, similarly, that $C_R$ exceeds $R$ by a substantial margin.

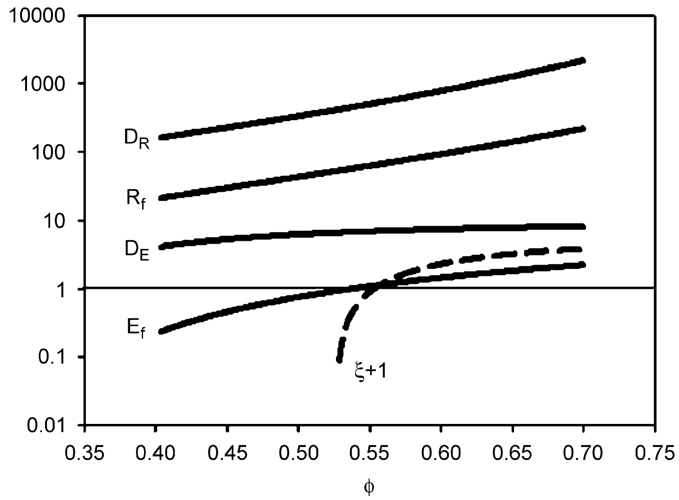


Fig. 2. Drag and stiffness parameters $R_f$ & $E_f$ and the derivatives $\partial R_f/\partial\phi$ & $\partial E_f/\partial\phi$ as functions of the solidosity $\phi$.

Table 3
Typical values for the static pore pressure and the secular correction at a vibration amplitude of the order of magnitude of the particle size

| | $p_{\mathrm{stat}}(0)$ | $A/\lambda_0$ |
|---|---|---|
| Magma problem | $2.6\times10^6$ Pa | $2\times10^5$ Pa |
| Filter problem | $10^3$ Pa | 130 Pa |

The secular contribution to the intergranular stress at $z=0$ due to the quadratic effect is (see Eqs. (28) and (27))

$$B-\frac{A}{2\lambda_0(1-\overline{\phi})}=\frac{\zeta\overline{R}\widehat{r}^2}{2\omega(1-\overline{\phi})^3}\left[\left(\frac{C_E}{E}-\frac{C_R}{\overline{R}}\right)(1-\overline{\phi})-2\right]. \tag{56}$$

Call the term in square brackets $\xi$ and this parameter is also plotted in Fig. 2. When $\xi>0$ then a positive secular stress follows, which would reduce the compressive intergranular stress from its static value. Note that this term depends on the frequency as $\omega^{-1}\widehat{r}^2$. It is furthermore observed that a small variation of the solidosity around the expected value of $\overline{\phi}\simeq 0.6$ may make this term either positive or negative. Add to that the fact that the solidosity-dependence of the resistance and the stiffness are rather approximately known and one can see that it will be very hard to predict whether a positive or negative extra stress is generated by the quadratic effect.

The other contribution to the stress is the one that follows from the linear effect—Eq. (23)—and this contribution takes the form

$$\overline{\pi}(0)=\frac{c\eta}{(1-\overline{\phi})}\frac{a}{h}\sqrt{\frac{\overline{R}\omega\zeta}{2E\overline{\phi}}}\widehat{r}, \tag{57}$$

which depends on the frequency as $\sqrt{\omega}\widehat{r}$.

The experiments reported in Ref. [1] indicate that the linear effect leads to the correct frequency dependence, suggesting that the quadratic effect is negligible at these solidosities.

For the pore pressures things are rather different, however. For this quantity there is no relevant linear contribution, but the influence of the quadratic contribution is outlined in Table 3 for a benchmark amplitude $\widehat{d}=a$.

It is seen that for the two sample problems at these amplitudes the pore pressure is reduced by some 10%. This occurs rapidly, so bubble formation may well occur, thus increasing the fluid compressibility, see Ref. [20]. This in turn has the effect of increasing $A/\lambda_0$, leading to a yet greater reduction and yet more bubbles.

## 4. Conclusion

Theory is presented that addresses the mechanical behaviour of an agitated particle–fluid mixture which is densely packed at a solids volume fraction of ca. 0.6. The problem is illustrated by two applications: (1) a sediment-filled magma chamber and (2) an oscillated dead-end filtration problem. For the latter experimental results are available, see Ref. [1]. The secular effects are calculated. These affect both the pore fluid pressure and the intergranular stress in the vicinity of the point at which the mixture is agitated. The key concepts are introduced, one of which is the particle pressure. This parameter may be associated with either collisions between particles or dissipative phenomena in a densely packed bed. The orders of magnitude for both are estimated, demonstrating that for amplitudes of vibration that are of the order of the particle diameter substantial effects will take place. These effects are chiefly the reduction of the pore fluid pressure and the reduction of the compressive solids-phase stress due to the linear particle pressure. The theory is essentially linear and when the secular effects become large, it is expected that major nonlinearities will have to be accounted for. The latter are not dealt with in this paper, which is intended to demonstrate the basic nature of an oscillated packed bed immersed in a fluid.

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